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A Joint Chance Constrained Programming with Bivariate Dagum Distribution

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Abstract A Joint chance-constrained programming (JCCP) technique is regarded as one of the most useful applicable techniques of stochastic programming techniques. It is more suitable for solving uncertain real problems, especially in economics and social problems, where some of model parameters are positive dependent random variables and follow well-known probability distributions. In this paper, we take into account a linear JCCP problem where some right-hand side random parameters are dependent and follow the Dagum distributions. So, firstly we derive a bivariate Dagum distribution with seven parameters with marginals following the Dagum distribution with three parameters. This proposed bivariate Dagum distribution is based on the Farlie-Gumbel-Morgensten copula (as presented in theorem (2.1)). Secondly, the proposed bivariate distribution is used in the context of JCCP technique to transform a linear JCCP model into an exact equivalent deterministic nonlinear programming model through theorem (3.1). Thirdly, through theorem (3.2), we prove that the obtained exact equivalent deterministic nonlinear programming model is a convex model, hence any nonlinear programming method can be used to solve it and find the global optimal solution. Finally, in order to demonstrate how to convert a linear JCCP model into an equivalent deterministic nonlinear programming model and solve it using the cutting plane method, a numerical example is included.

Keywords Farlie-Gumbel-Morgensten Copula, Chance Constrained Programming (CCP), JCCP, Bivariate Dagum Distribution, Multivariate Distributions, Equivalent Deterministic Model, Convex Model, Cutting Plane Method, Global Solution

1. Introduction

In 1977, Camilo Dagum first presented the three-parameter Dagum distribution for modelling personal income data [1]. The Pareto and Log-normal distributions can also be replaced with it [2, 3]. Additionally, several applications have used this distribution, for example, income and consumption data [4], financial data [5], and reliability analysis [6]. It is worth noting that the Dagum distribution is related to many distributions, for instance, the Burr type XII, Burr type III, Lomax, log logistic distribution, and the exponential distribution [2,7].

A continuous random variable is said to have a Dagum distribution, or $X \sim Dag(\beta;\lambda;\delta)$, if its probability density function (PDF), cumulative function (CDF), and the inverse cumulative function are respectively as follows [1,3]:

$$f(x;\beta;\lambda;\delta) = \beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}, \quad x>0, \quad \beta,\lambda,\delta>0, \tag{1}$$

$$F(x;\beta;\lambda;\delta) = \left(1+\lambda x^{-\delta}\right)^{-\beta}, \tag{2}$$

$$F^{-1}(y;\beta;\lambda;\delta)=\lambda^{\frac{1}{\delta}}\left(y^{-\frac{1}{\beta}}-1\right)^{-\frac{1}{\delta}},\qquad 0\le y\le 1, \tag{3}$$

where β and δ are the shape parameters, λ is the scale parameter, and y is the value of the cumulative function. Now, consider individual chance constraints having only one random parameter on the right-hand side (RHS), as follows:

$$P_r\left\{\sum_{j=1}^{n} a_{ij}x_j \ \ge \tilde{b}_i\right\}\ge \gamma_i \ \ , i=1,\dots,m_1, \tag{4}$$

$$P_r\left\{\sum_{j=1}^{n} a_{ij}x_j \ \le \tilde{b}_i\right\}\ge \gamma_i, i=m_1+1,\dots,m_2. \tag{5}$$

Here x_j , $j=1,2,\dots,n,$ represents the decision variable, $a_{ij}, i=1,2,\dots,m_2, j=1,2,\dots,n,$ is constant. and $\tilde{b}_i\sim Dag(\beta_i,\lambda_i,\delta_i), i=1,\dots,m_2$, Finally, $0\le\gamma_i\le 1$ represents the tolerance measure for the i^{th} constraint, then the exact equivalent deterministic linear constraints respectively are as follows [8]:

$$\sum_{j=1}^{n} a_{ij}x_j \ \ge \lambda_i^{\frac{1}{\delta_i}}\left(\gamma_i^{-\frac{1}{\beta_i}}-1\right)^{-\frac{1}{\delta_i}},\ i=1,\dots,m_1, \tag{6}$$

$$\sum_{j=1}^{n} a_{ij}x_j \ \le \lambda_i^{\frac{1}{\delta_i}}\left((1-\gamma_i)^{-\frac{1}{\beta_i}}-1\right)^{-\frac{1}{\delta_i}} \ ,i=m_1+1,\dots,m_2. \tag{7}$$

Recently, many bivariate Dagum distributions were presented. For example, in 2017, Muhammed [9] proposed a singular bivariate Dagum distribution using an idea proposed by Marshall and Olkin. In 2018, Popović et al [10] introduced a tractable bivariate extension of the univariate Dagum distribution by using the Marshall–Olkin approach and examining its dependence properties.

Historically the chance-constrained programming (CCP) technique was introduced by Charnas and Cooper [11] in 1959, in the case of individual chance constraints. Then, in 1965, Miller and Wagner [12] presented the joint chance constraints case. So, since the emergence of this technique, many studies have been presented, and most of those studies assumed the independence between random parameters for simplicity. However, usually many random parameters are dependent and follow non-negative distributions in real-world applications, especially in economic and social problems [8,13].

Therefore, many studies presented the equivalent deterministic programming models corresponding to the JCCP models under the dependence assumption. For example, in 2018, Hafez et al [14,15] assumed that the RHS or LHS parameters were dependent random variables (RVs) that followed the approximated Downton bivariate exponential distribution. In 2019, El-Dash [16] presented an extension of the Freund bivariate exponential distribution, assuming that the parameters in the RHS or some of the LHS parameters are RVs. Then, in 2020, El-Dash presented an extension of Farlie, Gumbel, and Morgenstern's bivariate exponential distribution [17]. In 2021, Ebaid and El-Dash [18] introduced a bivariate generalized exponential by the Cuadras-Augé copula function for the univariate generalized exponential distributions, assuming that the random parameters in the R.H.S. of joint constraints follow the proposed distribution. Then, in the same year, they assumed that the RHS parameters were dependent RVs that follow a derived bivariate Lomax distribution using the Farlie-Gumbel-Morgenstern (FGM) copula function [19].

The structure of this study is as follows: we present a new bivariate Dagum distribution using FGM Copula in Section 2. Following that, the JCCP linear model with dependent random parameters and its exact equivalent deterministic nonlinear programming model are provided in Section 3. We present a numerical example in Section 4. Finally, section 5 presents conclusions.

2. A new Bivariate Dagum Distribution Using FGM Copula

Copula functions are tools for modelling multivariate stochastic dependence between RVs and can be defined as functions that link or join multivariate distributions with their one-dimensional marginal functions. It was first introduced in the work of Sklar [20] in 1959 through his theorem. In literature, there are many families of copulas with one or more dependence parameters, for instance, the Farlie-Gumbel-Morgenstern copula, the Gaussian copula, the Archimedean copula, etc., and they have different properties that are indispensable in many applications [21]. One of the most important families is the FGM copula, which was introduced first by Morgenstern [22].

According to the bivariate FGM copula function, the joint CDF (JCDF) and the FGM copula density function for two RVs X_1, X_2 respectively, are defined by [16]:

$$F(x_1,x_2)=C(\mathrm{F}(x_1),\mathrm{G}(x_2))=\mathrm{F}(x_1)\mathrm{G}(x_2)[1+\theta(1-\mathrm{F}(x_1))(1-\mathrm{G}(x_2))],\qquad \theta\epsilon[-1,1], \tag{8}$$

$$c(\mathrm{F}(x_1),\mathrm{G}(x_2))=[1+\theta(1-2\mathrm{F}(x_1))(1-2\mathrm{G}(x_2))]. \tag{9}$$

Where: θ is the dependent parameter which shows the level of dependence between two RVs, so that when it is equal to zero, it means that the two variables are independent.

Also, the joint PDF (JPDF) is as follows:

$$f(x_1,x_2)=\mathrm{f}(x_1)\mathrm{g}(x_2)\cdot c(\mathrm{F}(x_1),\mathrm{G}(x_2)). \tag{10}$$

Through the following theorem, the FGM copula will be used to build a new bivariate Dagum distribution, where the JPDF and the JCDF are presented.

Theorem (2.1): Let $X_i \sim Dag(\beta_i;\lambda_i;\delta_i), i=1,2$, be dependent RVs, then the JCDF and the JPDF of (X_1, X_2) respectively are as follows:

$$F(x_1,x_2)=\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}+\theta\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\left\{1-\sum_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}+\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\right\}, \tag{11}$$

$$f(x_1,x_2)=\prod_{i=1}^{2}\beta_i\lambda_i\delta_i {x_i}^{-\delta_i-1}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i-1}\left[1+\theta\left(1-2\sum_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}+4\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\right)\right],$$
$$x_i>0,\quad \beta_i,\lambda_i,\delta_i>0,\ i=1,2,\quad -1\le\theta\le 1. \tag{12}$$

Proof: According to the bivariate FGM copula function, the JCDF is obtained as follows:

$$F(x_1,x_2)=\mathrm{F}(x_1)\mathrm{G}(x_2)[1+\theta(1-\mathrm{F}(x_1))(1-\mathrm{G}(x_2))]$$
$$=\left(1+\lambda_1 {x_1}^{-\delta_1}\right)^{-\beta_1}\left(1+\lambda_2 {x_2}^{-\delta_2}\right)^{-\beta_2}\left\{1+\theta\left[1-\left(1+\lambda_1 {x_1}^{-\delta_1}\right)^{-\beta_1}\right]\cdot\left[1-\left(1+\lambda_2 {x_2}^{-\delta_2}\right)^{-\beta_2}\right]\right\}$$
$$=\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\left\{1+\theta-\theta\sum_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}+\theta\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\right\}$$
$$=\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}+\theta\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\left\{1-\sum_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}+\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\right\}.$$

Then, the JPDF is obtained as follows:

$$f(x_1,x_2)=\mathrm{f}(x_1)\mathrm{g}(y)\cdot \mathrm{c}\ \left(\mathrm{F}(x_1),\mathrm{G}(y)\right)$$
$$=\beta_1\lambda_1\delta_1\beta_2\lambda_2\delta_2\cdot {x_1}^{-\delta_1-1}\left(1+\lambda_1 {x_1}^{-\delta_1}\right)^{-\beta_1-1}\cdot {x_2}^{-\delta_2-1}\left(1+\lambda_2 {x_2}^{-\delta_2}\right)^{-\beta_2-1}$$
$$\cdot\left\{1+\theta\left[1-2\left(1+\lambda_1 {x_1}^{-\delta_1}\right)^{-\beta_1}\right]\cdot\left[1-2\left(1+\lambda_2 {x_2}^{-\delta_2}\right)^{-\beta_2}\right]\right\}$$
$$=\prod_{i=1}^{2}\beta_i\lambda_i\delta_i {x_i}^{-\delta_i-1}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i-1}\cdot\left[1+\theta\left(1-2\sum_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}+4\prod_{i=1}^{2}\left(1+\lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i}\right)\right].$$

The surface plots of the bivariate PDF and CDF of the Dagum distribution (BDagum) for various parameter values $(\beta_1,\lambda_1,\delta_1,\beta_2,\lambda_2,\delta_2;\theta)$ are shown in Figures 1 and 2, respectively, as follows.

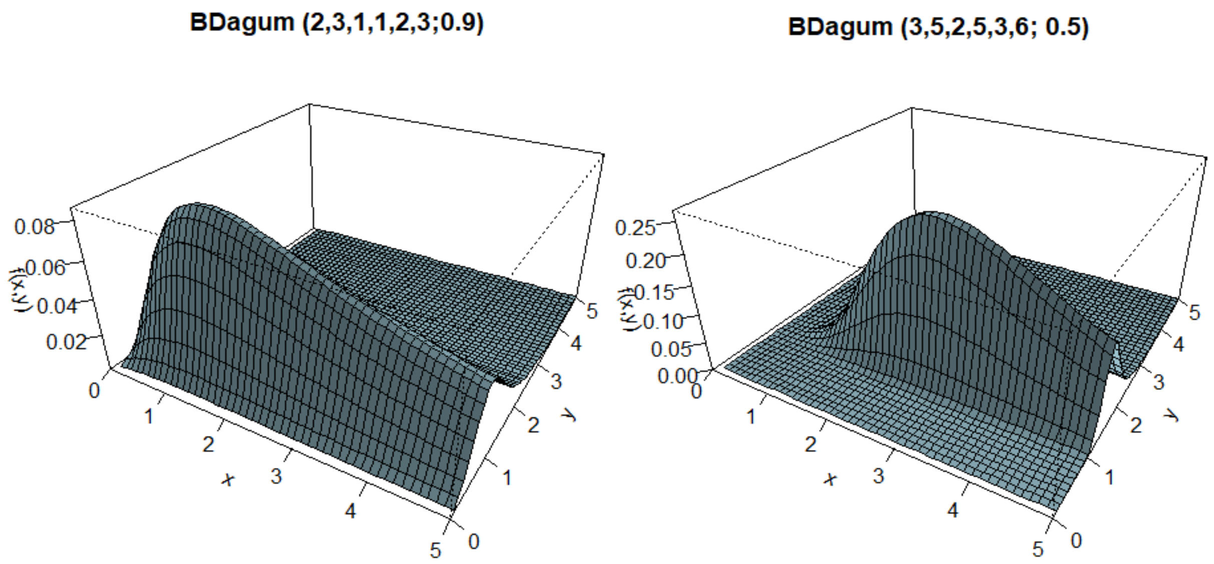

Figure 1. The JPDF plot of the BDagum for various values of parameters $(\beta_1, \lambda_1, \delta_1, \beta_2, \lambda_2, \delta_2; \theta)$.

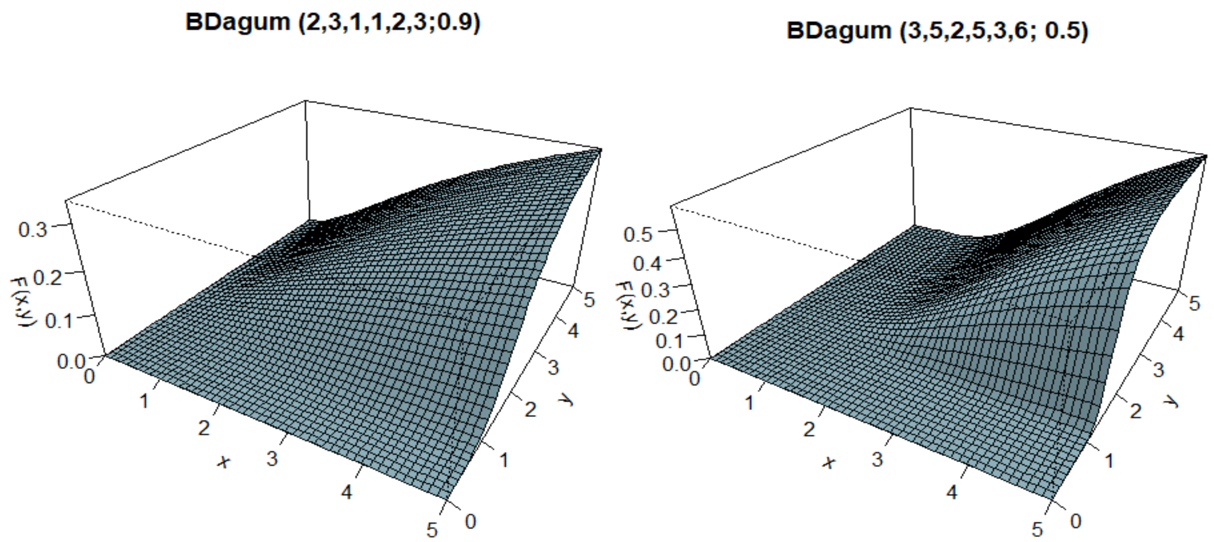

Figure 2. The JCDF plot of the BDagum for various values of parameters $(\beta_1, \lambda_1, \delta_1, \beta_2, \lambda_2, \delta_2; \theta)$.

Special Case (2.1): In case of independence between RVs $\boldsymbol{x_i, i = 1, 2}$, we have, $\boldsymbol{\theta = 0}$, and then the JCDF and the JPDF in (11) and (12), respectively, become as follows:

$$F(x_1, x_2) = \prod_{i=1}^{2} \left(1 + \lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i} \quad . \tag{13}$$

$$f(x_1, x_2) = \prod_{i=1}^{2} \beta_i \lambda_i \delta_i {x_i}^{-\delta_i - 1} \left(1 + \lambda_i {x_i}^{-\delta_i}\right)^{-\beta_i - 1}, \mathrm{x_i} > 0, \qquad \beta_i, \lambda_i, \delta_i > 0, \qquad i = 1,2. \tag{14}$$

In the next section, we will apply the previous theorem in the context of probabilistic programming model.

3. Joint Chance Constrained Linear Programming Model

Here, we will illustrate the conversion of the linear JCCP model into an exact equivalent deterministic nonlinear model assuming that some parameters follow the derived bivariate Dagum distribution [8,13].

Consider the following linear JCCP model:

$$Min. Z = \sum_{j=1}^{n} C_j \; x_j, \tag{15}$$

$$\text{S. T. } \sum_{j=1}^{n} a_{ij} x_j \; \geq b_i, \qquad i = 1,2, \dots, m, \tag{16}$$

$$P_r\left\{\sum_{j=1}^{n} a_{ij} x_j \; \geq \tilde{b}_i, \sum_{j=1}^{n} a_{i+1,j} x_j \; \geq \tilde{b}_{i+1}\right\} \geq \gamma_i \quad , \qquad i = m+1, \tag{17}$$

$$P_r\{\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i, \sum_{j=1}^{n} a_{i+1,j}x_j \le \tilde{b}_{i+1}\} \ge \gamma_i, \ \ i = m+2, \tag{18}$$

$$x_j \ge 0, \qquad j = 1, \dots, n, \tag{19}$$

where C_j, $j = 1, 2, \dots, n,$ and $b_i, i = 1, 2, \dots, m,$ are constants, and $(\tilde{b}_i, \tilde{b}_{i+1}) \sim BDag(\beta_i, \lambda_i, \delta_i, \beta_{i+1}, \lambda_{i+1}, \delta_{i+1}; \ \theta), i = m+1, m+2,$ are dependent random parameters and follow the above bivariate distribution in (11) and (12).

In the following theorem, we introduce the exact nonlinear equivalent deterministic constraints of the joint chance constraints in (17) and (18).

Theorem (3.1): Consider the joint chance constraints in (17) and (18). Then, the exact equivalent deterministic nonlinear constraints respectively are as follows:

$$\prod_{s=i}^{i+1} f_s(x_s)^{-\beta_s} + \theta \prod_{s=i}^{i+1} f_s(x_s)^{-\beta_s}\{1 - \sum_{s=i}^{i+1} f_s(x_s)^{-\beta_s} + \prod_{s=i}^{i+1} f_s(x_s)^{-\beta_s}\} \ge \gamma_i, \ \ i = m+1, \tag{20}$$

$$\prod_{s=i}^{i+1} f_s(x_s)^{-\beta_s} + \theta \prod_{s=i}^{i+1} f_s(x_s)^{-\beta_s}\{1 - \sum_{s=i}^{i+1} f_s(x_s)^{-\beta_s} + \prod_{s=i}^{i+1} f_s(x_s)^{-\beta_s}\} \le (1-\gamma_i), \ \ i = m+2, \tag{21}$$

Where $f_s(x_s) = \left(1 + \lambda_s\left(\sum_{j=1}^{n} a_{sj}x_j\right)^{-\delta_s}\right)$.

Proof: For the joint chance constraints in (17), it is clear that [8]:

$$P_r\left\{\sum_{j=1}^{n} a_{ij}x_j \ge \tilde{b}_i, \sum_{j=1}^{n} a_{i+1,j}x_j \ge \tilde{b}_{i+1}\right\} \ge \gamma_i \quad \rightarrow \quad F_{\tilde{b}_i,\tilde{b}_{i+1}}\left(\sum_{j=1}^{n} a_{ij}x_j, \sum_{j=1}^{n} a_{i+1,j}x_j\right) \ge \gamma_i,$$

where $F_{\tilde{b}_i,\tilde{b}_{i+1}}(.,.)$ represents the JCDF of $\tilde{b}_i, \tilde{b}_{i+1}$ defined in (11). Therefore, by direct substitution, we get constraint (20).

Similarly, for the joint chance constraints in (18) we have,

$$P_r\left\{\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i, \sum_{j=1}^{n} a_{i+1,j}x_j \le \tilde{b}_{i+1}\right\} \ge \gamma_i \quad \rightarrow F_{\tilde{b}_i,\tilde{b}_{i+1}}\left(\sum_{j=1}^{n} a_{ij}x_j, \sum_{j=1}^{n} a_{i+1,j}x_j\right) \le (1-\gamma_i),$$

Therefore, by substituting $F_{\tilde{b}_i,\tilde{b}_{i+1}}(\dots)$, we get constraint (21).

Special case (3.1): When $\boldsymbol{\theta = 0}$, then the corresponding exact equivalent deterministic linear constraints of the constraints in (20) and (21) respectively are as follows:

$$0 \le \sum_{j=1}^{n} a_{ij}x_j \le A_i, \text{ and } 0 \le \sum_{j=1}^{n} a_{i+1,j}x_j \le A_{i+1}, \tag{22}$$

$$0 \le \sum_{j=1}^{n} a_{ij}x_j \le B_i, \text{ and } 0 \le \sum_{j=1}^{n} a_{i+1,j}x_j \le B_{i+1}, \tag{23}$$

where the constants are $A_i = (\lambda_i)^{\frac{1}{\delta_i}}\left(e^{\frac{-\ln(\gamma_i)}{\beta_i}} - 1\right)^{-\frac{1}{\delta_i}}$, $A_{i+1} = (\lambda_{i+1})^{\frac{1}{\delta_{i+1}}}\left(e^{\frac{-\ln(\gamma_{i+1})}{\beta_{i+1}}} - 1\right)^{-\frac{1}{\delta_{i+1}}}$,

$$B_i = (\lambda_i)^{\frac{1}{\delta_i}}\left(e^{\frac{-\ln(1-\gamma_i)}{\beta_i}} - 1\right)^{-\frac{1}{\delta_i}}, \ B_{i+1} = (\lambda_{i+1})^{\frac{1}{\delta_{i+1}}}\left(e^{\frac{-\ln(1-\gamma_{i+1})}{\beta_{i+1}}} - 1\right)^{-\frac{1}{\delta_{i+1}}}.$$

Proof: In the case of independence, then $\boldsymbol{\theta = 0}$ in constraint (20) and the equivalent deterministic nonlinear constraint is as follows:

$$\left[1 + \lambda_i\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{-\delta_i}\right]^{-\beta_i} \cdot \left[1 + \lambda_{i+1}\left(\sum_{j=1}^{n} a_{i+1,j}x_j\right)^{-\delta_{i+1}}\right]^{-\beta_{i+1}} \ge \gamma_i.$$

The previous constraint is a nonlinear constraint, a logarithmic transformation could be applied as follows:

$$-\beta_i ln\left[1 + \lambda_i\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{-\delta_i}\right] - \beta_{i+1} ln\left[1 + \lambda_{i+1}\left(\sum_{j=1}^{n} a_{i+1,j}x_j\right)^{-\delta_{i+1}}\right] \ge \ln(\gamma_i).$$

Let $h_i = ln\left[1 + \lambda_i\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{-\delta_i}\right]$ and $h_{i+1} = ln\left[1 + \lambda_{i+1}\left(\sum_{j=1}^{n} a_{i+1,j}x_j\right)^{-\delta_{i+1}}\right]$ (24)

$$\therefore \beta_i h_i + \beta_{i+1} h_{i+1} \le -\ln(\gamma_i).$$

Therefore,

$$0 \le h_i \le \frac{-\ln(\gamma_i)}{\beta_i} \quad \text{and } 0 \le h_{i+1} \le \frac{-\ln(\gamma_i)}{\beta_{i+1}}. \tag{25}$$

Using (24), (25) we get

$$\sum_{j=1}^{n} a_{ij}x_j = (\lambda_i)^{\frac{1}{\delta_i}}(e^{h_i}-1)^{-\frac{1}{\delta_i}} = (\lambda_i)^{\frac{1}{\delta_i}}\left(e^{\frac{-\ln(\gamma_i)}{\beta_i}}-1\right)^{-\frac{1}{\delta_i}}.$$

Hence

$$0 \le \sum_{j=1}^{n} a_{ij}x_j \le A_i, \tag{26}$$

Also,

$$0 \le \sum_{j=1}^{n} a_{i+1,j}x_j \le A_{i+1}, \tag{27}$$

where $A_i = (\lambda_i)^{\frac{1}{\delta_i}}\left(e^{\frac{-\ln(\gamma_i)}{\beta_i}}-1\right)^{-\frac{1}{\delta_i}}, A_{i+1} = (\lambda_{i+1})^{\frac{1}{\delta_{i+1}}}\left(e^{\frac{-\ln(\gamma_{i+1})}{\beta_{i+1}}}-1\right)^{-\frac{1}{\delta_{i+1}}}$

Similarly, for constraint (21), the exact equivalent deterministic linear constraint is as follows:

$$0 \le \sum_{j=1}^{n} a_{ij}x_j \le B_i, \tag{28}$$

Also,

$$0 \le \sum_{j=1}^{n} a_{i+1,j}x_j \le B_{i+1} \tag{29}$$

where $B_i = (\lambda_i)^{\frac{1}{\delta_i}}\left(e^{\frac{-\ln(1-\gamma_i)}{\beta_i}}-1\right)^{-\frac{1}{\delta_i}}, B_{i+1} = (\lambda_{i+1})^{\frac{1}{\delta_{i+1}}}\left(e^{\frac{-\ln(1-\gamma_{i+1})}{\beta_{i+1}}}-1\right)^{-\frac{1}{\delta_{i+1}}}.$

Theorem (3.2): Consider the nonlinear function in the LHS for constraint (20) or (21). Then the exact equivalent deterministic nonlinear programming model in (15)-(16) and (19)-(21) is a convex model.

Proof: The model in (15) -(16) and (19) -(21) is a convex model if and only if the LHS of constraints (20) or (21) are strictly convex functions, which occurs if the following condition is met [23]:

$$F[\alpha x^{(1)} + (1-\alpha)x^{(2)}] < \alpha\, F(x^{(1)}) + (1-\alpha)F(x^{(2)})$$

Then let $x^{(1)} = \binom{X_1}{X_2}$, $x^{(2)} = \binom{X_1+\Delta_1}{X_2+\Delta_2}$, $x^{(1)} \ne x^{(2)}$, $x^{(1)}, x^{(2)} \ge 0$,

Now, by substituting the LHS of either constraint (20) or (21)- because both constraints have the same LHS, into the LHS of the previous condition, then,

$$\text{LHS} = F[\alpha\ x^{(1)} + (1-\alpha)x^{(2)}] = F\left[\alpha\binom{X_1}{X_2} + (1-\alpha)\binom{X_1+\Delta_1}{X_2+\Delta_2}\right] = F\left[\binom{\alpha X_1}{\alpha X_2} + \binom{(1-\alpha)(X_1+\Delta_1)}{(1-\alpha)(X_2+\Delta_2)}\right] = F\binom{X_1-\alpha\Delta_1+\Delta_1}{X_2-\alpha\Delta_2+\Delta_2}$$

$$= (1+\theta)\left(1+\frac{\lambda_1}{(X_1-\alpha\Delta_1+\Delta_1)^{\delta_1}}\right)^{-\beta_1}\left(1+\frac{\lambda_2}{(X_2-\alpha\Delta_2+\Delta_2)^{\delta_2}}\right)^{-\beta_2} - \theta\left(1+\frac{\lambda_1}{(X_1-\alpha\Delta_1+\Delta_1)^{\delta_1}}\right)^{-\beta_1}\left(1+\frac{\lambda_2}{(X_2-\alpha\Delta_2+\Delta_2)^{\delta_2}}\right)^{-2\beta_2} - \theta\left(1+\frac{\lambda_1}{(X_1-\alpha\Delta_1+\Delta_1)^{\delta_1}}\right)^{-2\beta_1}\left(1+\frac{\lambda_2}{(X_2-\alpha\Delta_2+\Delta_2)^{\delta_2}}\right)^{-\beta_2} + \theta\left(1+\frac{\lambda_1}{(X_1-\alpha\Delta_1+\Delta_1)^{\delta_1}}\right)^{-2\beta_1}\cdot\left(1+\frac{\lambda_2}{(X_2-\alpha\Delta_2+\Delta_2)^{\delta_2}}\right)^{-2\beta_2}.$$

Using the binomial series**,** where $(1+x)^{\varphi} = \sum_{n=0}^{\infty} C_n^{\varphi} x^n$, φ is real number, $|x| < 1$,

$$\text{LHS} = \Big\{(1+\theta)\sum_{n=0}^{\infty} C_n^{-\beta_1}\sum_{m=0}^{\infty} C_m^{-\beta_2} + \theta\sum_{n=0}^{\infty} C_n^{-2\beta_1}\sum_{m=0}^{\infty} C_m^{-2\beta_2} - \theta\sum_{n=0}^{\infty} C_n^{-\beta_1}\sum_{m=0}^{\infty} C_m^{-2\beta_2} - \theta\sum_{n=0}^{\infty} C_n^{-2\beta_1}\sum_{m=0}^{\infty} C_m^{-\beta_2}\Big\}\left\{\left[\frac{\lambda_1}{(X_1-\alpha\Delta_1+\Delta_1)^{\delta_1}}\right]^n\left[\frac{\lambda_2}{(X_2-\alpha\Delta_2+\Delta_2)^{\delta_2}}\right]^m\right\}, \tag{30}$$

Similarly, by substituting the LHS of constraint (20) or (21) into the RHS in the convexity condition, then:

$$\text{RHS} = \alpha F\left(x^{(1)}\right) + (1-\alpha)F\left(x^{(2)}\right) = \alpha\ f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + (1-\alpha)\ f\begin{pmatrix} (X_1+\Delta_1) \\ (X_2+\Delta_2) \end{pmatrix}$$

$$= \alpha\left\{(1+\theta)\left(1+\frac{\lambda_1}{X_1{}^{\delta_1}}\right)^{-\beta_1}\left(1+\frac{\lambda_2}{X_2{}^{\delta_2}}\right)^{-\beta_2} - \theta\left(1+\frac{\lambda_1}{X_1}\right)^{-\beta_1}\left(1+\frac{\lambda_2}{X_2}\right)^{-2\beta_2} - \theta\left(1+\frac{\lambda_1}{X_1{}^{\delta_1}}\right)^{-2\beta_1}\left(1+\frac{\lambda_2}{X_2{}^{\delta_2}}\right)^{-\beta_2} + \theta\left(1+\frac{\lambda_1}{X_1{}^{\delta_1}}\right)^{-2\beta_1}\left(1+\frac{\lambda_2}{X_2{}^{\delta_2}}\right)^{-2\beta_2}\right\}$$

$$+ (1-\alpha)\left\{(1+\theta)\cdot\left(1+\frac{\lambda_1}{(X_1+\Delta_1)^{\delta_1}}\right)^{-\beta_1}\left(1+\frac{\lambda_2}{(X_2+\Delta_2)^{\delta_2}}\right)^{-\beta_2} - \theta\left(1+\frac{\lambda_1}{(X_1+\Delta_1)}\right)^{-\beta_1}\cdot\left(1+\frac{\lambda_2}{(X_2+\Delta_2)}\right)^{-2\beta_2} - \theta\left(1+\frac{\lambda_1}{(X_1+\Delta_1)^{\delta_1}}\right)^{-2\beta_1}\left(1+\frac{\lambda_2}{(X_2+\Delta_2)^{\delta_2}}\right)^{-\beta_2} + \theta\left(1+\frac{\lambda_1}{(X_1+\Delta_1)^{\delta_1}}\right)^{-2\beta_1}\left(1+\frac{\lambda_2}{(X_2+\Delta_2)^{\delta_2}}\right)^{-2\beta_2}\right\}.$$

Similarly using the binomial series, we get

$$\text{RHS} = \left\{(1+\theta)\sum_{n=0}^{\infty} C_n^{-\beta_1}\sum_{m=0}^{\infty} C_m^{-\beta_2} + \theta\sum_{n=0}^{\infty} C_n^{-2\beta_1}\sum_{m=0}^{\infty} C_m^{-2\beta_2} - \theta\sum_{n=0}^{\infty} C_n^{-\beta_1}\cdot\sum_{m=0}^{\infty} C_m^{-2\beta_2} - \theta\sum_{n=0}^{\infty} C_n^{-2\beta_1}\sum_{m=0}^{\infty} C_m^{-\beta_2}\right\}\left\{\alpha\left[\frac{\lambda_1}{X_1{}^{\delta_1}}\right]^n\left[\frac{\lambda_2}{X_2{}^{\delta_2}}\right]^m + \left[\frac{\lambda_1}{(X_1+\Delta_1)^{\delta_1}}\right]^n\left[\frac{\lambda_2}{(X_2+\Delta_2)^{\delta_2}}\right]^m - \boldsymbol{\alpha}\left[\frac{\lambda_1}{(X_1+\Delta_1)^{\delta_1}}\right]^n\left[\frac{\lambda_2}{(X_2+\Delta_2)^{\delta_2}}\right]^m\right\}. \quad (31)$$

From (30) and (31), it is clear that the condition is satisfied.

Thus, the function on the LHS of the constraint (20) or (21) is a strictly convex function. So, the model in (15)-(16), (19)-(21) is a strictly convex model. Also, this theorem satisfies the case of independence in the previous special case. This theorem is important because the convex model can be solved using the cutting plane method and get the global optimal solution [23,24].

4. A Numerical Example

Consider the following JCCP model:

$$\text{Min.}\quad Z = 2x_1 + x_2, \quad (32)$$

$$\text{S. T.}\ 2x_1 + x_2 \geq 6, \quad (33)$$

$$x_1 + x_2 \leq 5, \quad (34)$$

$$P_r\{4x_1 + 2x_2 \geq \widetilde{b_3}, 2x_1 + x_2 \geq \widetilde{b_4}\} \geq 0.9, \quad (35)$$

$$x_j \geq 0, j = 1,2, \quad (36)$$

where $(\widetilde{b_3}, \widetilde{b_4}) \sim BDag(\beta_1 = 1, \lambda_1 = 2, \delta_1 = 2, \beta_2 = 2, \lambda_2 = 1, \delta_2 = 2, \theta = 0.8)$ are dependent random parameters and follow the derived bivariate Dagum distribution.

According to theorem (3.1), the exact equivalent deterministic nonlinear programming model using the JCCP technique, which corresponds to the above model, is as follows:

$$\text{Min.}\quad Z = 2x_1 + x_2, \quad (37)$$

$$\text{S. T.}\ 2x_1 + x_2 \geq 6, \quad (38)$$

$$x_1 + x_2 \leq 5, \quad (39)$$

$$\frac{1.8\left(1+2(4x_1+2x_2)^{-2}\right)^{-1}}{(1+(\ 2x_1+x_2)^{-2})^2} - \frac{0.8\left(1+2(4x_1+2x_2)^{-2}\right)^{-1}}{(1+(\ 2x_1+x_2)^{-2})^4} - \frac{0.8\left(1+2(4x_1+2x_2)^{-2}\right)^{-2}}{(1+(\ 2x_1+x_2)^{-2})^2} + \frac{0.8\left(1+2(4x_1+2x_2)^{-2}\right)^{-2}}{(1+(\ 2x_1+x_2)^{-2})^4} \geq 0.9, \quad (40)$$

The above model is convex according to theorem (3.2), therefore it can be solved by the cutting plane method [12,22], assuming that the bounds of the decision variables are

$$0.1 \leq x_1 \leq 4,\ 0.1 \leq x_2 \leq 3 \quad (41)$$

Then, the global optimal solution is as follows:

$$x_1 = 2.95, x_2 = 0.1, z^* = 6$$

5. Conclusions

Throughout this paper, the JPDF and the JCDF of the new bivariate Dagum distribution are presented through theorem (2.1) by using the FGM copula. Then, the JCCP technique is used to transform JCCP model into an exact nonlinear equivalent deterministic programming model through theorem (3.1), where some parameters in the JCCP linear model follow the derived bivariate Dagum distribution, then a special case is introduced regarding independent parameters. After that, we have proved that the obtained exact nonlinear equivalent deterministic programming model is a convex model through theorem (3.2) and a global optimal solution could be obtained. Finally, a numerical example is presented.

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